Improved q-exponential and q-trigonometric functions

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Abstract

We propose a new definition of the q-exponential function. Our q-exponential function maps the imaginary axis into the unit circle and the resulting q-trigonometric functions are bounded and satisfy the Pythagorean identity.

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1 Introduction

The quantum calculus (q-calculus) is an old, classical branch of mathematics, which can be traced back to Euler and Gauss [11, 21] with important contributions of Jackson a century ago [18, 19]. In recent years there are many new developments and applications of the q-calculus in mathematical physics, especially concerning special functions [1, 8, 12, 14] and quantum mechanics [4, 5, 25, 13, 10, 23, 29]. Many papers were devoted to various approaches to q-deformations of elementary functions, including exponential and trigonometric functions [2, 3, 7, 15, 22, 24, 26, 27, 28].

In this paper we propose new definitions of the q-exponential function and q-trigonometric functions. These results are motivated by recent developments in the time scales calculus, where new exponential, hyperbolic and trigonometric function have been defined [9]. The concept of time scales unifies difference and differential calculus [16]. The q-calculus can be considered as a calculus on a special time scale (see, e.g., [6]).

The functions presented in this paper have better qualitative properties than standard q-exponential and q-trigonometric functions. In order to discuss and compare these properties we begin with a short summary of the classical results, usually following the textbook [20].

In the standard approach to the q-calculus two exponential function are used:

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \ , \qquad E_q^z = \sum_{n=1}^{\infty} \frac{z^n}{[\tilde{n}]!} \ , \tag{1}$$

where q is positive, z is complex, and

$$\begin{aligned} &[n]! = [1][2]\ldots[n] \ , \qquad [k] = 1 + q + q^2 + \ldots + q^{k-1} \ , \\ &[\tilde{n}]! = [\tilde{1}][\tilde{2}]\ldots[\tilde{n}] \ , \qquad [\tilde{k}] = 1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^{k-1}} \ . \end{aligned} \tag{2}$$

Hence we immediately get $E_q^z = e_{1/q}^z$. Another, more popular, form of E_q^z is obtained using the identity

$$[\tilde{n}]! = q^{\frac{(1-n)n}{2}} [n]! \ . \tag{3}$$

Both exponential functions can be represented by infinite products,

$$e_q^z = \prod_{k=0}^{\infty} (1 - (1-q)q^k z)^{-1}, \qquad E_q^z = \prod_{k=0}^{\infty} (1 + (1-q)q^k z) \ . \tag{4}$$

From this form we easily see that $e_q^z E_q^{-z} = 1$. Moreover,

$$D_q e_q^z = e_q^z \ , \qquad D_q E_q^z = E_q^{qz} \ , \tag{5}$$

where D_q (q-derivative or Jackson's derivative) is defined by

$$D_q f(z) := \frac{f(qz) - f(q)}{qz - z} \ . \tag{6}$$

The existence of two representations of q-exponential functions (infinite series and infinite product) is related to well known formulae for the usual exponential function ($q = 1$),

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \lim_{m\to\infty} \left(1 + \frac{z}{m}\right)^m \ . \tag{7}$$

Two exponential functions of the quantum calculus generate two pairs of the q-trigonometric functions. Using notation of [20] we have:

$$\begin{aligned} \sin_q x &= \frac{e_q^{ix} - e_q^{-ix}}{2i} \ , \quad \mathrm{Sin}_q x = \frac{E_q^{ix} - E_q^{-ix}}{2i} \ , \\ \cos_q x &= \frac{e_q^{ix} + e_q^{-ix}}{2} \ , \quad \mathrm{Cos}_q x = \frac{E_q^{ix} + E_q^{-ix}}{2} \ . \end{aligned} \tag{8}$$

Taking into account properties of q-exponential functions (see above) we easily derive properties of standard q-trigonometric functions:

$$\begin{aligned} &\cos_q x \ \mathrm{Cos}_q x + \sin_q x \ \mathrm{Sin}_q x = 1 \ , \\ &\sin_q x \ \mathrm{Cos}_q x = \cos_q x \ \mathrm{Sin}_q x \ , \end{aligned} \tag{9}$$

$$\begin{aligned} &D_q \sin_q x = \cos_q x \ , \quad D_q \cos_q x = -\sin_q x \ , \\ &D_q \mathrm{Sin}_q x = \mathrm{Cos}_q (qx) \ , \quad D_q \mathrm{Cos}_q x = -\mathrm{Sin}_q (qx) \ . \end{aligned} \tag{10}$$

Note that the corresponding tangents coincide: $\mathrm{Tan}_q x = \tan_q x$.

2 Improved q-exponential function

New q-exponential function $\mathcal{E}_q^z$ is defined as

$$\mathcal{E}_q^z := e_q^{\frac{z}{2}} \, E_q^{\frac{z}{2}} = \prod_{k=0}^{\infty} \frac{1 + q^k (1-q)\frac{z}{2}}{1 - q^k (1-q)\frac{z}{2}} \ , \tag{11}$$

where e_q^z, E_q^z are standard q-exponential functions. This definition is motivated by the classical Cayley transformation

$$z \to \operatorname{cay}(z, a) := \frac{1 + az}{1 - az} \ , \tag{12}$$

see, e.g., [9, 17]. Indeed,

$$\mathcal{E}_q^{qz} = \frac{1 - (1-q)\frac{z}{2}}{1 + (1-q)\frac{z}{2}} \, \mathcal{E}_q^z = \operatorname{cay}\left(-\frac{z}{2}, \ 1-q\right) \ \mathcal{E}_q^z \ . \tag{13}$$

Theorem 1. *The q-exponential function $\mathcal{E}_q^z$ is analytic in the disc $|z| < R_q$ and*

$$\mathcal{E}_q^z = \sum_{n=0}^{\infty} \frac{z^n}{\{n\}!} \ , \tag{14}$$

for $|z| < R_q$, where

$$R_q = \begin{cases} \dfrac{2}{1-q} & \textit{for} \quad 0 < q < 1 \ , \\[2ex] \dfrac{2q}{q-1} & \textit{for} \quad q > 1 \ , \\[2ex] \infty & \textit{for} \quad q = 1 \ , \end{cases} \tag{15}$$

$$\{n\} := \frac{1 + q + \ldots + q^{n-1}}{\frac{1}{2}(1 + q^{n-1})} = \frac{[n]}{\frac{1}{2}(1 + q^{n-1})} = \frac{2(1 - q^n)}{(1-q)(1 + q^{n-1})} \ , \tag{16}$$

and, finally, $\{n\}! = \{1\}\{2\} \ldots \{n\}$.

Proof: In the disc $|z| < 1$ both series (1) are absolutely convergent for any $q \in \mathbb{R}_+$. Multiplying them we get

$$e_q^{\frac{z}{2}} \, E_q^{\frac{z}{2}} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}} \left(\frac{z}{2}\right)^{k+j}}{[k]![j]!} = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^n}{[n]!} \left(\sum_{j=0}^{n} \frac{q^{\frac{j(j-1)}{2}} [n]!}{[j]![n-j]!} \right) \ . \tag{17}$$

Using Gauss's binomial formula (see, e.g., [20], formula (5.5))

$$\prod_{j=0}^{n-1} (z + a q^j) = \sum_{j=0}^{n} \frac{q^{\frac{j(j-1)}{2}} \ [n]!}{[j]![n-j]!} \ a^j z^{n-j} \ , \tag{18}$$

we have, as a particular case,

$$\sum_{j=0}^{n} \frac{q^{\frac{j(j-1)}{2}} \ [n]!}{[j]![n-j]!} = (1+1)(1+q) \ldots (1+q^{n-1}) \ . \tag{19}$$

Substituting (19) into (17) we get the formula (14) with $\{n\}$ defined by (16). In order to obtain the radius of convergence, we compute

$$\lim_{n\to\infty} \left| \frac{z^{n+1}}{\{n+1\}!} \right| \left| \frac{\{n\}!}{z^n} \right| = \lim_{n\to\infty} \left| \frac{z}{\{n+1\}} \right| = \begin{cases} \frac{(1-q)|z|}{2} & \text{for} \quad q < 1 \\ \frac{(q-1)|z|}{2q} & \text{for} \quad q > 1 \end{cases} \tag{20}$$

Then, using d'Alembert's test, we get (for $q \neq 1$) the radius of convergence (15). Note that $R_{1/q} = R_q$. In the case $q = 1$ all q-exponential functions coincide with e^z, hence $R_1 = \infty$. □

Theorem 2. *The q-exponential function $\mathcal{E}_q^z$ has the following properties:*

$$\mathcal{E}_q^{-z} = \left(\mathcal{E}_q^z\right)^{-1} \ , \qquad | \ \mathcal{E}_q^{ix}| = 1 \ , \tag{21}$$

$$\mathcal{E}_q^z = \mathcal{E}_{1/q}^z \ , \qquad D_q \mathcal{E}_q^z = \langle \mathcal{E}_q^z \rangle \ , \tag{22}$$

where $z \in \mathbb{C}$, $x \in \mathbb{R}$ and we use the notation $\langle f(z) \rangle := \dfrac{f(z) + f(qz)}{2}$.

Proof: The first equation of (21) is a straightforward consequence of the definition (11). Then, $\overline{\mathcal{E}_q^z} = \mathcal{E}_q^{\bar{z}}$. Hence,

$$|\mathcal{E}_q^{ix}|^2 = \overline{\mathcal{E}_q^{ix}} \mathcal{E}_q^{ix} = \mathcal{E}_q^{-ix} \mathcal{E}_q^{ix} = 1 \ . \tag{23}$$

The symbol $\{n\}$ depends on q. In this proof it is convenient to use more precise notation $\{n\} \equiv \{n\}_q$, $\{n\}! \equiv \{n\}_q!$. The equation $\mathcal{E}_q^z = \mathcal{E}_{1/q}^z$ follows immediately from the obvious identity

$$\{n\}_q! = \{n\}_{1/q}! \ . \tag{24}$$

Finally,

$$D_q \mathcal{E}_q^z = \frac{\mathcal{E}_q^{qz} - \mathcal{E}_q^z}{qz - z} = \frac{\mathcal{E}_q^z}{(q-1)z} \left(\frac{1 - (1-q)\frac{z}{2}}{1 + (1-q)\frac{z}{2}} - 1 \right) = \frac{\mathcal{E}_q^{qz}}{1 + (1-q)\frac{z}{2}} \ , \tag{25}$$

$$\langle \mathcal{E}_q^z \rangle = \frac{1}{2} \left(\mathcal{E}_q^{qz} + \mathcal{E}_q^z \right) = \frac{1}{2} \left(\frac{1 - (1-q)\frac{z}{2}}{1 + (1-q)\frac{z}{2}} + 1 \right) \mathcal{E}_q^z = \frac{\mathcal{E}_q^{qz}}{1 + (1-q)\frac{z}{2}} \ , \tag{26}$$

which implies the second equation of (22). □

The properties (21) are identical with analogical properties of the exponential function e^z. We point out that neither e_q^z nor E_q^z satisfies (21). Instead, we have $E_q^{-z} e_q^z = 1$.

3 Improved q-trigonometric functions

New q-sine and q-cosine functions are defined in a natural way:

$$\mathcal{S}in_q x = \frac{\mathcal{E}_q^{ix} - \mathcal{E}_q^{-ix}}{2i} \ , \quad \mathcal{C}os_q x = \frac{\mathcal{E}_q^{ix} + \mathcal{E}_q^{-ix}}{2} \ . \tag{27}$$

Theorem 3. *q-Trigonometric functions defined by (27) satisfy:*

$$\begin{aligned} &\mathcal{C}os_q^2 x + \mathcal{S}in_q^2 x = 1 \ , \\ &D_q \mathcal{S}in_q x = \langle \mathcal{C}os_q x \rangle \ , \\ &D_q \mathcal{C}os_q x = -\langle \mathcal{S}in_q x \rangle \ , \end{aligned} \tag{28}$$

Proof: Properties (28) follow directly from (21), (22) (note that $\mathcal{E}_q^{ix} \mathcal{E}_q^{-ix} = 1$). □

Corollary 4. *q-Trigonometric functions $\mathcal{C}os_q x$, $\mathcal{S}in_q x$ are real for $x \in \mathbb{R}$. Moreover, for any $x \in \mathbb{R}$, we have*

$$-1 \leqslant \mathcal{C}os_q x \leqslant 1 \ , \qquad -1 \leqslant \mathcal{S}in_q x \leqslant 1 \ . \tag{29}$$

Theorem 5. *New q-trigonometric functions can be expressed by standard q-trigonometric functions as follows:*

$$
\begin{aligned}
\mathcal{C}os_q 2x &= \cos_q x \ \mathrm{Cos}_q x - \sin_q x \ \mathrm{Sin}_q x = \frac{1-\tan_q^2 x}{1+\tan_q^2 x} \ , \\
\mathcal{S}in_q 2x &= \sin_q x \ \mathrm{Cos}_q x + \cos_q x \ \mathrm{Sin}_q x = \frac{2\tan_q x}{1+\tan_q^2 x} \ .
\end{aligned}
\tag{30}
$$

Proof: First, we compute

$$
\begin{aligned}
\cos_q x \ \mathrm{Cos}_q x - \sin_q x \ \mathrm{Sin}_q x &= \frac{e_q^{ix} E_q^{ix} + e_q^{-ix} E_q^{-ix}}{2} = \mathcal{C}os_q 2x \ , \\
\sin_q x \ \mathrm{Cos}_q x + \cos_q x \ \mathrm{Sin}_q x &= \frac{e_q^{ix} E_q^{ix} - e_q^{-ix} E_q^{-ix}}{2} = \mathcal{S}in_q 2x \ .
\end{aligned}
\tag{31}
$$

Then, using (9), we get

$$
\begin{aligned}
\mathcal{C}os_q 2x &= \frac{\cos_q x \ \mathrm{Cos}_q x - \sin_q x \ \mathrm{Sin}_q x}{\cos_q x \ \mathrm{Cos}_q x + \sin_q x \ \mathrm{Sin}_q x} = \frac{1 - \tan_q x \ \mathrm{Tan}_q x}{1 - \tan_q x \ \mathrm{Tan}_q x} \ , \\
\mathcal{S}in_q 2x &= \frac{\sin_q x \ \mathrm{Cos}_q x + \cos_q x \ \mathrm{Sin}_q x}{\cos_q x \ \mathrm{Cos}_q x + \sin_q x \ \mathrm{Sin}_q x} = \frac{\tan_q x + \mathrm{Tan}_q x}{1 - \tan_q x \ \mathrm{Tan}_q x} \ .
\end{aligned}
\tag{32}
$$

Taking into account $\mathrm{Tan}_q x = \tan_q x$ we complete the proof. □

4 Conclusions

Motivated by the classical Cayley transformation and recent results in the time scales calculus (see [9]), we introduced a new definition of the q-exponential function. Main advantages of the new q-exponential function consist in better qualitative properties (i.e., its properties are more similar to properties of e^z). In particular, it maps the unitary axis into the unit circle, compare (21), which implies excellent properties of new trigonometric functions, including formulae (28) and boundedness (29).

Especially interesting is the Pythagorean identity: $\mathcal{C}os_q^2 x + \mathcal{S}in_q^2 x = 1$. According to our best knowledge, other q-defomations of trigonometric functions do not satisfy this property. The same concerns even the paper [15], full of surprising identities.

Our exponential function is closely related to both popular q-exponential functions (1). Therefore, proofs and calculations concerning $\mathcal{E}_q^z$ can be usually done with the help of known results. We plan to express in terms of the new exponential function classical results containing q-exponential functions, and we hope to obtain some improvements.

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